NAG Fortran Library Routine Document

D05AAF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

D05AAF solves a linear, non-singular Fredholm equation of the second kind with a split kernel.

2 Specification

```
 SUBROUTINE D05AAF(LAMBDA, A, B, K1, K2, G, F, C, N, IND, W1, W2, WD,
1                  NMAX, MN, IFAIL)
 INTEGER           N, IND, NMAX, MN, IFAIL
 real              LAMBDA, A, B, K1, K2, G, F(N), C(N), W1(NMAX,MN),
1                  W2(MN,4), WD(MN)
 EXTERNAL          K1, K2, G
```

3 Description

D05AAF solves an integral equation of the form

$$f(x) - \lambda \int_a^b k(x,s) f(s)\, ds = g(x)$$

for $a \leq x \leq b$, when the kernel k is defined in two parts: $k = k_1$ for $a \leq s \leq x$ and $k = k_2$ for $x < s \leq b$. The method used is that of El-Gendi (1969) for which, it is important to note, each of the functions k_1 and k_2 must be defined, smooth and non-singular, for all x and s in the interval $[a, b]$.

An approximation to the solution $f(x)$ is found in the form of an n term Chebyshev-series $\sum_{i=1}^{n}{}' c_i T_i(x)$, where $'$ indicates that the first term is halved in the sum. The coefficients c_i, for $i = 1, 2, .., n$, of this series are determined directly from approximate values f_i, for $i = 1, 2, \ldots, n$, of the function $f(x)$ at the first n of a set of $m + 1$ Chebyshev points:

$$x_i = \tfrac{1}{2}(a + b + (b - a)\cos[(i - 1)\pi/m]), \quad i = 1, 2, \ldots, m + 1.$$

The values f_i are obtained by solving simultaneous linear algebraic equations formed by applying a quadrature formula (equivalent to the scheme of Clenshaw and Curtis (1960)) to the integral equation at the above points.

In general $m = n - 1$. However, if the kernel k is centro-symmetric in the interval $[a, b]$, i.e., if $k(x, s) = k(a + b - x, a + b - s)$, then the routine is designed to take advantage of this fact in the formation and solution of the algebraic equations. In this case, symmetry in the function $g(x)$ implies symmetry in the function $f(x)$. In particular, if $g(x)$ is even about the mid-point of the range of integration, then so also is $f(x)$, which may be approximated by an even Chebyshev-series with $m = 2n - 1$. Similarly, if $g(x)$ is odd about the mid-point then $f(x)$ may be approximated by an odd series with $m = 2n$.

4 References

Clenshaw C W and Curtis A R (1960) A method for numerical integration on an automatic computer *Numer. Math.* **2** 197–205

El-Gendi S E (1969) Chebyshev solution of differential, integral and integro-differential equations *Comput. J.* **12** 282–287

5 Parameters

1: LAMBDA – ***real*** *Input*

On entry: the value of the parameter λ of the integral equation.

2: A – ***real*** *Input*

On entry: the lower limit of integration, a.

3: B – ***real*** *Input*

On entry: the upper limit of integration, b.

Constraint: B > A.

4: K1 – ***real*** FUNCTION, supplied by the user. *External Procedure*

K1 must evaluate the kernel $k(x,s) = k_1(x,s)$ of the integral equation for $a \leq s \leq x$.

Its specification is:

```
real FUNCTION K1(X, S)
real             X, S
```

1: X – ***real*** *Input*
2: S – ***real*** *Input*

On entry: the values of x and s at which $k_1(x,s)$ is to be evaluated.

K1 must be declared as EXTERNAL in the (sub)program from which D05AAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

5: K2 – ***real*** FUNCTION, supplied by the user. *External Procedure*

K2 must evaluate the kernel $k(x,s) = k_2(x,s)$ of the integral equation for $x < s \leq b$.

Its specification is:

```
real FUNCTION K2(X, S)
real             X, S
```

1: X – ***real*** *Input*
2: S – ***real*** *Input*

On entry: the values of x and s at which $k_2(x,s)$ is to be evaluated.

K2 must be declared as EXTERNAL in the (sub)program from which D05AAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

Note that the functions k_1 and k_2 must be defined, smooth and non-singular for all x and s in the interval $[a,b]$.

6: G – ***real*** FUNCTION, supplied by the user. *External Procedure*

G must evaluate the function $g(x)$ for $a \leq x \leq b$.

Its specification is:

```
real FUNCTION G(X)
real          X
```

1: X – ***real*** *Input*

On entry: the values of x at which $g(x)$ is to be evaluated.

G must be declared as EXTERNAL in the (sub)program from which D05AAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

7: F(N) – ***real*** array *Output*

On exit: the approximate values f_i, for $i = 1,2,...,\mathrm{N}$ of $f(x)$ evaluated at the first N of $\mathrm{M}+1$ Chebyshev points x_i, (see Section 3).

If IND is 0 or 3, $\mathrm{M} = \mathrm{N} - 1$; if IND is 1, $\mathrm{M} = 2 \times \mathrm{N}$ and if IND is 2, $\mathrm{M} = 2 \times \mathrm{N} - 1$.

8: C(N) – ***real*** array *Output*

On exit: the coefficients c_i, for $i = 1,2,...,\mathrm{N}$ of the Chebyshev-series approximation to $f(x)$.

If IND is 1 this series contains polynomials of odd order only and if IND is 2 the series contains even order polynomials only.

9: N – INTEGER *Input*

On entry: the number of terms in the Chebyshev-series required to approximate $f(x)$.

10: IND – INTEGER *Input*

On entry: IND must be set to 0,1,2 or 3.

IND = 0

$k(x,s)$ is not centro-symmetric (or no account is to be taken of centro-symmetry).

IND = 1

$k(x,s)$ is centro-symmetric and $g(x)$ is odd.

IND = 2

$k(x,s)$ is centro-symmetric and $g(x)$ is even.

IND = 3

$k(x,s)$ is centro-symmetric but $g(x)$ is neither odd nor even.

11: W1(NMAX,MN) – ***real*** array *Workspace*
12: W2(MN,4) – ***real*** array *Workspace*
13: WD(MN) – ***real*** array *Workspace*

14: NMAX – INTEGER *Input*

On entry: the first dimension of the array W1 as declared in the (sub)program from which D05AAF is called.

Constraint: $\mathrm{NMAX} \geq \mathrm{N}$.

15: MN – INTEGER *Input*

On entry: the first dimension of the array W2 as declared in the (sub)program from which D05AAF is called.

Constraint: $\mathrm{MN} \geq 2 \times \mathrm{N} + 2$.

16: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

On exit: IFAIL $= 0$ unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, for users not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL $= 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL $= 1$

On entry, A $\geq$ B.

IFAIL $= 2$

A failure has occurred (in F04AAF unless N $= 1$) due to proximity to an eigenvalue. In general, if LAMBDA is near an eigenvalue of the integral equation, the corresponding matrix will be nearly singular.

7 Accuracy

No explicit error estimate is provided by the routine but it is usually possible to obtain a good indication of the accuracy of the solution either

(i) by examining the size of the later Chebyshev coefficients c_i, or

(ii) by comparing the coefficients c_i or the function values f_i for two or more values of N.

8 Further Comments

The time taken by the routine increases with N.

This routine may be used to solve an equation with a continuous kernel by calling the same FUNCTION for K2 as for K1.

This routine may also be used to solve a Volterra equation by defining K2 (or K1) to be identically zero.

9 Example

The example program solves the equation

$$f(x) - \int_0^1 k(x,s)f(s)\,ds = (1 - \frac{1}{\pi^2})\sin(\pi x)$$

where

$$k(x,s) = \begin{cases} s(1-x) & \text{for} \quad 0 \leq s < x, \\ x(1-s) & \text{for} \quad x \leq s \leq 1. \end{cases}$$

Five terms of the Chebyshev-series are sought, taking advantage of the centro-symmetry of the $k(x,s)$ and even nature of $g(x)$ about the mid-point of the range $[0,1]$.

The approximate solution at the point $x = 0.1$ is calculated by calling C06DBF.

9.1 Program Text

Note: the listing of the example program presented below uses ***bold italicised*** terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     D05AAF Example Program Text
*     Mark 14 Revised.  NAG Copyright 1989.
*     .. Parameters ..
      INTEGER          N, NMAX, MN
      PARAMETER        (N=5,NMAX=N,MN=2*N+2)
      INTEGER          NOUT
      PARAMETER        (NOUT=6)
*     .. Scalars in Common ..
      real             R
*     .. Local Scalars ..
      real             A, ANS, B, LAMBDA, X
      INTEGER          I, IFAIL, IND, IS
*     .. Local Arrays ..
      real             C(NMAX), F(NMAX), W1(NMAX,MN), W2(MN,4), WD(MN)
*     .. External Functions ..
      real             C06DBF, G, K1, K2, X01AAF
      EXTERNAL         C06DBF, G, K1, K2, X01AAF
*     .. External Subroutines ..
      EXTERNAL         D05AAF
*     .. Common blocks ..
      COMMON           R
*     .. Executable Statements ..
      WRITE (NOUT,*) 'D05AAF Example Program Results'
      WRITE (NOUT,*)
      R = X01AAF(0.0e0)
      LAMBDA = 1.0e0
      A = 0.0e0
      B = 1.0e0
      IND = 2
      IFAIL = 0
      WRITE (NOUT,*)
     +'Kernel is centro-symmetric and G is even so the solution is even'
      WRITE (NOUT,*)
*
      CALL D05AAF(LAMBDA,A,B,K1,K2,G,F,C,N,IND,W1,W2,WD,NMAX,MN,IFAIL)
*
      WRITE (NOUT,*) 'Chebyshev coefficients'
      WRITE (NOUT,*)
      WRITE (NOUT,99998) (C(I),I=1,N)
      WRITE (NOUT,*)
      X = 0.1e0
*     Note that X has to be transformed to range [-1,1]
      IS = 1
      IF (IND.EQ.1) THEN
         IS = 3
      ELSE
         IF (IND.EQ.2) IS = 2
      END IF
      ANS = C06DBF(2.0e0/(B-A)*(X-0.5e0*(B+A)),C,N,IS)
      WRITE (NOUT,99999) 'X=', X, '    ANS=', ANS
      STOP
*
99999 FORMAT (1X,A,F5.2,A,1F10.4)
99998 FORMAT (1X,5e14.4)
      END
*
      real   FUNCTION K1(X,S)
*     .. Scalar Arguments ..
      real             S, X
*     .. Executable Statements ..
      K1 = S*(1.0e0-X)
      RETURN
      END
*
      real   FUNCTION K2(X,S)
```

```
*     .. Scalar Arguments ..
      real              S, X
*     .. Executable Statements ..
      K2 = X*(1.0e0-S)
      RETURN
      END
*
      real  FUNCTION G(X)
*     .. Scalar Arguments ..
      real              X
*     .. Scalars in Common ..
      real              R
*     .. Intrinsic Functions ..
      INTRINSIC         SIN
*     .. Common blocks ..
      COMMON            R
*     .. Executable Statements ..
      G = SIN(R*X)*(1.0e0-1.0e0/(R*R))
      RETURN
      END
```

9.2 Program Data

None.

9.3 Program Results

```
 D05AAF Example Program Results

 Kernel is centro-symmetric and G is even so the solution is even

 Chebyshev coefficients

     0.9440E+00   -0.4994E+00    0.2799E-01   -0.5967E-03    0.6658E-05

 X= 0.10    ANS=    0.3090
```